Construction of an Algebraic Closure
Math 581 Fall, 1998

In this note we give a construction of an algebraic closure of an arbitrary field. This construction is due to Emil Artin. Zorn's lemma is not invoked in this proof, unlike the one given in class. We do indirectly use Zorn's lemma since we require the existence of maximal ideals inside arbitrary commutative rings with identity, which does require Zorn's lemma. However, we avoid the use of cardinal arithmetic.

Theorem 1 *Let F be a field. Then there exists an algebraic closure of F.*

Proof. Suppose we have constructed extension fields $F \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that for all n, (1) F_{n+1} is algebraic over F_n, and (2) every nonconstant $f(x) \in F_n[x]$ has a root in F_{n+1}. Let K be the union of the F_n for all n. Since the $\{F_n\}$ form an ascending chain of fields, K is a field extension of F. Also, since being algebraic is a transitive property, we see that every F_n is algebraic over F, so K is also algebraic over F. We claim that K is algebraically closed, which will then show K is an algebraic closure of F. To see this, take a nonconstant $g(x) \in K[x]$. Since there are only finitely many coefficients of $g(x)$, we see that $g(x) \in F_n[x]$ for some n. Then $g(x)$ has a root in $F_{n+1} \subseteq K$. This shows that K is algebraically closed.

We now inductively construct the fields F_n. Suppose that $F \subseteq F_1 \subseteq \cdots \subseteq F_n$ have been constructed. We then construct F_{n+1}. Consider the polynomial ring $F_n[\{x_f\}]$ in the variables x_f, one variable for each monic irreducible polynomial $f(x) \in F_n[x]$. Let I be the ideal of $F_n[\{x_f\}]$ generated by $\{f(x_f)\}$ for each such f. Then $I \neq F_n[\{x_f\}]$, which is proved in the lemma below. Set $R = F_n[\{x_f\}]/I$, a nonzero ring. Let M be a maximal ideal of R, and finally set $F_{n+1} = R/M$. We have a sequence of ring homomorphisms

$$F_n \longrightarrow F_n[\{x_f\}] \longrightarrow F_n[\{x_f\}]/I = R \longrightarrow R/M = F_{n+1}.$$

Since the map $F_n \to F_{n+1}$ is not the zero map (as $1 \mapsto 1$), this map is $1-1$, so we may assume $F_n \subseteq F_{n+1}$. The ring $F_{n+1} = R/M$ is a field since M is a maximal ideal of R. Furthermore, since the last two maps above are onto, if a_f is the image in F_{n+1} of x_f then $F_{n+1} = F_n[\{a_f\}]$. Now $f(x_f) \to 0$ in R since $f(x_f) \in I$. Thus $f(x_f) \to 0$ in F_{n+1}. But $f(x_f) \to f(a_f)$, so $f(a_f) = 0$. This shows that each a_f is algebraic over F_n. Hence as F_{n+1} is generated over F_n by the a_f, F_{n+1} is algebraic over F_n. Finally, if $g(x) \in F_n[x]$ then let $f(x)$ be a monic irreducible factor of $g(x)$. Then $f(x)$ has a root in F_{n+1}, namely a_f. Thus $g(x)$ has a root in F_{n+1}. This completes the proof of the properties of F_{n+1}. ∎

The last step in the proof is to demonstrate that the ideal I defined above is nonzero. We do this in the following lemma.

Lemma 2 *Let F be a field and $F[\{x_i\}]$ the polynomial ring in the variables x_i, where i ranges over some set $\mathcal{I}$. Suppose for each i that $f_i(x)$ is a monic irreducible polynomial over F. Then the ideal I of $F[\{x_i\}]$ generated by $f_i(x_i)$ for each i is a proper ideal.*

Proof. Suppose $I = F[\{x_i\}]$. Then there is an n and polynomials $g_1, \ldots, g_n$ such that

$$1 = f_1(x_{i_1})g_1(x_{i_1}, \ldots, x_{i_n}) + \cdots + f_n(x_{i_n})g_n(x_{i_1}, \ldots, x_{i_n}).$$

For simplicity we shall write x_m in place of x_{i_m} for each m. We can assume that all the g_m involve only the variables $x_1, \ldots, x_n$ by increasing the number of f_m if necessary in an equation of this type. Suppose n is chosen to be minimal such that we have such an expression involving n of the x_i. If $S = F[x_1, \ldots, x_n]$ then $(f_1(x_1), \ldots, f_n(x_n)) = S$. Let $R = F[x_1, \ldots, x_{n-1}]$. By minimality of n we have $(f_1(x_1), \ldots, f_{n-1}(x_{n-1})) \neq R$. Let us view the above equation as taking place in $S = R[x_n]$. If $c_m = f_m(x_m) \in R$ we have $J = (c_1, \ldots, c_{n-1}, f_n(x_n)) = S$. Now set $I_0 = (c_1, \ldots, c_{n-1}) \subseteq R$. So $J = (I, f_n(x_n))$. There are ring homomorphisms

$$R[x_n] \longrightarrow (R/I_0)[x_n] \longrightarrow \frac{(R/I_0)[x_n]}{(\overline{f_n(x_n)})}$$

where $\overline{f_n(x_n)}$ is the image of $f_n(x_n)$ in $(R/I_0)[x]$. Since R/I_0 is a nonzero ring and $\overline{f_n(x_n)}$ is not a unit (as f_n is monic of degree at least 1) we see that this last ring is nonzero. Hence the kernel of the composite homomorphism is a proper ideal of S. But J lies in this kernel, so $J \neq S$. This contradiction shows our original I is a proper ideal of $F[\{x_i\}]$, proving the lemma. ∎