Construction of an Algebraic Closure
Math 581 Fall, 1998

In this note we give a construction of an algebraic closure of an arbitrary field. This construction is due to Emil Artin. Zorn’s lemma is not invoked in this proof, unlike the one given in class. We do indirectly use Zorn’s lemma since we require the existence of maximal ideals inside arbitrary commutative rings with identity, which does require Zorn’s lemma. However, we avoid the use of cardinal arithmetic.

**Theorem 1** Let $F$ be a field. Then there exists an algebraic closure of $F$.

**Proof.** Suppose we have constructed extension fields $F \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that for all $n$, (1) $F_{n+1}$ is algebraic over $F_n$, and (2) every nonconstant $f(x) \in F_n[x]$ has a root in $F_{n+1}$. Let $K$ be the union of the $F_n$ for all $n$. Since the $\{F_n\}$ form an ascending chain of fields, $K$ is a field extension of $F$. Also, since being algebraic is a transitive property, we see that every $F_n$ is algebraic over $F$, so $K$ is also algebraic over $F$. We claim that $K$ is algebraically closed, which will then show $K$ is an algebraic closure of $F$. To see this, take a nonconstant $g(x) \in K[x]$. Since there are only finitely many coefficients of $g(x)$, we see that $g(x) \in F_n[x]$ for some $n$. Then $g(x)$ has a root in $F_{n+1} \subseteq K$. This shows that $K$ is algebraically closed.

We now inductively construct the fields $F_n$. Suppose that $F \subseteq F_1 \subseteq \cdots \subseteq F_n$ have been constructed. We then construct $F_{n+1}$. Consider the polynomial ring $F_n[\{x_f\}]$ in the variables $x_f$, one variable for each monic irreducible polynomial $f(x) \in F_n[x]$. Let $I$ be the ideal of $F_n[\{x_f\}]$ generated by $\{f(x_f)\}$ for each such $f$. Then $I \neq F_n[\{x_f\}]$, which is proved in the lemma below. Set $R = F_n[\{x_f\}]/I$, a nonzero ring. Let $M$ be a maximal ideal of $R$, and finally set $F_{n+1} = R/M$. We have a sequence of ring homomorphisms

$$F_n \rightarrow F_n[\{x_f\}] \rightarrow F_n[\{x_f\}]/I = R \rightarrow R/M = F_{n+1}.$$  

Since the map $F_n \rightarrow F_{n+1}$ is not the zero map (as $1 \mapsto 1$), this map is $1 - 1$, so we may assume $F_n \subseteq F_{n+1}$. The ring $F_{n+1} = R/M$ is a field since $F_n$ is a maximal ideal of $R$. Furthermore, since the last two maps above are onto, if $a_f$ is the image in $F_{n+1}$ of $x_f$ then $F_{n+1} = F_n[\{a_f\}]$. Now $f(x_f) \rightarrow 0$ in $R$ since $f(x_f) \in I$. Thus $f(x_f) \rightarrow 0$ in $F_{n+1}$. But $f(x_f) \rightarrow a_f$, so $a_f$ is a root of $f(x_f)$. This shows that each $a_f$ is algebraic over $F_n$. Hence as $F_{n+1}$ is generated over $F_n$ by the $a_f$, $F_{n+1}$ is algebraic over $F_n$. Finally, if $g(x) \in F_n[x]$ then let $f(x)$ be a monic irreducible factor of $g(x)$. Then $f(x)$ has a root in $F_{n+1}$, namely $a_f$. Thus $g(x)$ has a root in $F_{n+1}$. This completes the proof of the properties of $F_{n+1}$. □

The last step in the proof is to demonstrate that the ideal $I$ defined above is nonzero. We do this in the following lemma.

**Lemma 2** Let $F$ be a field and $F[\{x_i\}]$ the polynomial ring in the variables $x_i$, where $i$ ranges over some set $I$. Suppose for each $i$ that $f_i(x)$ is a monic irreducible polynomial over $F$. Then the ideal $I$ of $F[\{x_i\}]$ generated by $f_i(x_i)$ for each $i$ is a proper ideal.

**Proof.** Suppose $I = F[\{x_i\}]$. Then there is an $n$ and polynomials $g_1, \ldots, g_n$ such that

$$1 = f_1(x_i_1)g_1(x_i_1, \ldots, x_i_n) + \cdots + f_n(x_i_n)g_n(x_i_1, \ldots, x_i_n).$$

For simplicity we shall write $x_m$ in place of $x_m$ for each $m$. We can assume that all the $g_m$ involve only the variables $x_1, \ldots, x_n$ by increasing the number of $f_m$ if necessary in an equation of this type. Suppose $n$ is chosen to be minimal such that we have such an expression involving $n$ of the $x_i$. If $S = F[x_1, \ldots, x_n]$ then $(f_1(x_1), \ldots, f_n(x_n)) = S$. Let $R = F[x_1, \ldots, x_{n-1}]$. By minimality of $n$ we have $(f_1(x_1), \ldots, f_{n-1}(x_{n-1})) \neq R$. Let us view the above equation as taking place in $S = R[x_n]$. If $c_m = f_m(x_m) \in R$ we have $J = (c_1, \ldots, c_{n-1}, f_n(x_n)) = S$. Now set $I_0 = (c_1, \ldots, c_{n-1}) \subseteq R$. So $J = (I, f_n(x_n))$. There are ring homomorphisms

$$R[x_n] \rightarrow (R/I_0)[x_n] \rightarrow (R/I_0)[x_n]/(f_n(x_n))$$

where $f_n(x_n)$ is the image of $f_n(x_n)$ in $(R/I_0)[x_n]$. Since $R/I_0$ is a nonzero ring and $f_n(x_n)$ is not a unit (as $f_n$ is monic of degree at least 1) we see that this last ring is nonzero. Hence the kernel of the composite homomorphism is a proper ideal of $S$. But $J$ lies in this kernel, so $J \neq S$. This contradiction shows our original $I$ is a proper ideal of $F[\{x_i\}]$, proving the lemma. □