The (Bessel, Jacobi, Laguerre, Legendre)-type linear fourth-order differential equations: remarks on history and special functions

W.N. Everitt

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Measure theory

- Let $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be monotonic non-decreasing on $\mathbb{R}$; then $\hat{\mu}$ generates a non-negative Baire measure μ on the σ-algebra $\mathcal{B}$ of Borel sets of $\mathbb{R}$. In turn μ defines the Lebesgue-Stieltjes, Hilbert function space $L^2(\mathbb{R};\mu)$ with inner-product

$$(f,g)_\mu := \int_{\mathbb{R}} f\overline{g}\ d\mu = \int_{\mathbb{R}} f(x)\overline{g}(x)\ d\mu(x)$$

for all equivalence classes generated by $f, g \in L^2(\mathbb{R};\mu)$.

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- If $\hat{\mu} \in AC_{\mathrm{loc}}(\mathbb{R})$, with respect to Lebesgue measure, then the derivative $\hat{\mu}'$ exists almost everywhere and is non-negative on $\mathbb{R}$. In this case if $f, g \in L^2(\mathbb{R}; \mu)$ then f, g belong to the weighted Lebesgue Hilbert function space $L^2(\mathbb{R} : \hat{\mu}')$ and

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- Throughout suppose that $\hat{\mu}$ is not constant on $\mathbb{R}$.

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- Given the Hilbert function space $L^2(\mathbb{I};\mu)$ let $\hat{\mu}$ be chosen so that the moment condition

$$\int_{\mathbb{I}} |x|^n \ d\mu(x) < +\infty \text{ for all } n \in \mathbb{N}_0$$

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  is satisfied.
- Given the functions $x \longmapsto x^n \in L^2(\mathbb{I};\mu)$ for all $n \in \mathbb{N}_0$ suppose that for all $n \in \mathbb{N}_0$ the result

$$\int_{\mathbb{I}} \left| \sum_{r=0}^{n} \alpha_r x^r \right|^2 d\mu(x) = 0,$$

  with $\{\alpha_r \in \mathbb{C} : r = 0, 1, \ldots, n\}$, implies that $\alpha_r = 0$ for $r = 0, 1, \ldots, n$.

- An application of the Gram-Schmidt orthogonalisation process gives a sequence $\{\varphi_n : n \in \mathbb{N}_0\}$ of real-valued, linearly independent polynomials each in the Hilbert space $L^2(\mathbb{I};\mu)$ and respectively of degree exactly n, with the orthogonal property

$$\int_{\mathbb{I}} \varphi_m(x)\varphi_n(x)\ d\mu(x) = 0,$$

for all $m, n \in \mathbb{N}_0$ with $m \neq n$. Also

$$k_n := \int_{\mathbb{I}} |\varphi_n(x)|^2\ d\mu(x) > 0 \text{ for all } n \in \mathbb{N}_0.$$

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- The set $\{\varphi_n : n \in \mathbb{N}_0\}$ is complete in the space $L^2(\mathbb{I};\mu)$ if the support of the measure μ is a bounded set on the real line $\mathbb{R}$; if the support of μ is not bounded then the set may or may not be complete.

- An application of the Gram-Schmidt orthogonalisation process gives a sequence $\{\varphi_n : n \in \mathbb{N}_0\}$ of real-valued, linearly independent polynomials each in the Hilbert space $L^2(\mathbb{I};\mu)$ and respectively of degree exactly n, with the orthogonal property

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- The set $\{\varphi_n : n \in \mathbb{N}_0\}$ is complete in the space $L^2(\mathbb{I};\mu)$ if the support of the measure μ is a bounded set on the real line $\mathbb{R}$; if the support of μ is not bounded then the set may or may not be complete.
- For all $n \in \mathbb{N}_0$, the n zeros of φ_n are all real, simple and contained in $\mathbb{I}$; these zeros interlace the $n+1$ zeros of φ_{n+1}.

Let the independent parameters $\alpha, \beta \in \mathbb{R}$ and satisfy the condition $\alpha, \beta > -1$. (Legendre polynomials is a special case with $\alpha = \beta = 0$.) Define $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\mu}(x) := \begin{cases} -\int_{-1}^{0} (1-t)^{\alpha}(1+t)^{\beta}\ dt & \text{for all } x \in (-\infty, -1) \\ \int_{0}^{x} (1-t)^{\alpha}(1+t)^{\beta}\ dt & \text{for all } x \in [-1, +1] \\ \int_{0}^{1} (1-t)^{\alpha}(1+t)^{\beta}\ dt & \text{for all } x \in (+1, +\infty). \end{cases}$$

Then $\hat{\mu}$ is monotonic non-decreasing on $\mathbb{R}$ and $\hat{\mu} \in AC_{\mathrm{loc}}(\mathbb{R})$. The measure μ generated by $\hat{\mu}$ has compact support $[-1, +1]$ and the weighted Lebesgue Hilbert function space $L^2(\mathbb{R} : \hat{\mu}')$ has the inner-product

$$(f, g)_{\mu} = \int_{-1}^{+1} f(x)\overline{g}(x)(1-x)^{\alpha}(1+x)^{\beta}\ dx.$$

The associated Jacobi orthogonal polynomials are complete in the weighted space $L^2((-1, +1); (1-x)^{\alpha}(1+x)^{\beta})$, and are denoted by

$$\left\{ P_n^{(\alpha,\beta)}(x) : \text{for all } x \in [-1, +1] \text{ and all } n \in \mathbb{N}_0 \right\}.$$

Let the independent parameter $\alpha \in \mathbb{R}$ and satisfy the condition $\alpha > -1$. Define $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\mu}(x) := \begin{cases} 0 & \text{for all } x \in (-\infty, 0) \\ \int_0^x t^\alpha \exp(-t)\ dt & \text{for all } x \in [0, +\infty) \end{cases}$$

Then $\hat{\mu}$ is monotonic non-decreasing on $\mathbb{R}$ and $\hat{\mu} \in AC_{\text{loc}}(\mathbb{R})$. The measure μ generated by $\hat{\mu}$ has support $[0, +\infty)$ and the weighted Lebesgue Hilbert function space $L^2(\mathbb{R} : \hat{\mu}')$ has the inner-product

$$(f, g)_\mu = \int_0^{+\infty} f(x)\overline{g}(x) x^\alpha \exp(-x)\ dx.$$

The associated Laguerre orthogonal polynomials are complete in the weighted space $L^2((0, +\infty); x^\alpha \exp(-x))$, and are denoted by

$$\left\{ L_n^\alpha(x) : \text{for all } x \in [0, +\infty) \text{ and all } n \in \mathbb{N}_0 \right\}.$$

Hermite polynomials

Define $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\mu}(x) := \int_0^x \exp(-t^2)\ dt \text{ for all } x \in (-\infty, +\infty)$$

Then $\hat{\mu}$ is monotonic non-decreasing on $\mathbb{R}$ and $\hat{\mu} \in AC_{\text{loc}}(\mathbb{R})$. The measure μ generated by $\hat{\mu}$ has support $(-\infty, +\infty)$ and the weighted Lebesgue Hilbert function space $L^2(\mathbb{R} : \hat{\mu}')$ has the inner-product

$$(f, g)_\mu = \int_{-\infty}^{+\infty} f(x)\overline{g}(x) \exp(-x^2)\ dx.$$

The associated Hermite orthogonal polynomials are complete in the weighted space $L^2((-\infty, +\infty); \exp(-x^2))$, and are denoted by

$$\{H_n(x) : \text{for all } x \in (-\infty, +\infty) \text{ and all } n \in \mathbb{N}_0\}.$$

Sturm-Liouville theory

Given the open interval $(a, b) \subseteq \mathbb{R}$ let the three coefficients $p, q, w : (a, b) \to \mathbb{R}$ and satisfy the conditions

$$p^{-1}, q, w \in L^1_{\text{loc}}(a, b) \text{ and } p, w > 0 \text{ almost everywhere on } (a, b).$$

Then the associated Sturm-Liouville differential equation is

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a, b),$$

where $\lambda \in \mathbb{C}$ is the complex-valued spectral parameter.
The spectral properties of this equation are studied in the weighted Hilbert function space $L^2((a, b); w)$.
The GKN theory of symmetric boundary conditions applied to the endpoints a and b, yields a continuum of self-adjoint differential operators, say $\{T\}$, in the space $L^2((a, b); w)$, all generated by the expression $w^{-1}(-(pf')' + qf)$. Here the elements f in the domain $D(T)$ of a particular operator T, have to satisfy certain differentiability properties and the chosen symmetric boundary conditions.

Differential equations

Sturm-Liouville differential equations for Jacobi, Laguerre, Hermite cases.

- **Jacobi**

  In this case the differential equation is, for all $x \in (-1, +1)$,

  $$-((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x))' = \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x).$$

  Eigenvalues $\lambda_n = n(n+\alpha+\beta+1)$ for all $n \in \mathbb{N}_0$. Eigensolutions $P_n^{(\alpha,\beta)}(x)$.

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  Eigenvalues $\lambda_n = n(n+\alpha+\beta+1)$ for all $n \in \mathbb{N}_0$. Eigensolutions $P_n^{(\alpha,\beta)}(x)$.

- **Laguerre**

  In this case the differential equation is, for all $x \in (0, +\infty)$

  $$-(x^{\alpha+1}\exp(-x)y'(x))' = \lambda x^{\alpha}\exp(-x)y(x).$$

  Eigenvalues $\lambda_n = n$ for all $n \in \mathbb{N}_0$. Eigensolutions $L_n^{\alpha}(x)$.

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  Eigenvalues $\lambda_n = n$ for all $n \in \mathbb{N}_0$. Eigensolutions $L_n^{\alpha}(x)$.
- **Hermite**
  In this case the differential equation is, for all $x \in (-\infty, +\infty)$,
  $$-(\exp(-x^2)y'(x))' = \lambda\exp(-x^2)y(x).$$
  Eigenvalues $\lambda_n = 2n$ for all $n \in \mathbb{N}_0$. Eigensolutions $H_n(x)$.

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3. Eigenvectors of right-definitive, self-adjoint Sturm-Liouville differential operators.
4. In his paper of 1929 Bochner proved that the only sets of orthogonal polynomials, defined on intervals of the real line $\mathbb{R}$, that are generated by Sturm-Liouville differential equations are the Jacobi, Laguerre and Hermite polynomials. For this reason and for historical connections, these Jacobi, Laguerre and Hermite polynomials are named the classical orthogonal polynomials.

Krall

In two papers, written in 1938 and 1940, H.L. Krall solved the problem of finding all fourth-order Lagrange symmetric (formally self-adjoint) fourth-order differential equations, generating sets of orthogonal polynomials on intervals of the real line $\mathbb{R}$. He found six sets of such polynomials:

- Three sets are from the formal squares of the three classical differential expressions and reproduce the Jacobi, Laguerre and Hermite orthogonal polynomials on their respective intervals of $\mathbb{R}$.

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  Legendre-type, Jacobi-type, Laguerre-type.

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- There is no Hermite-type differential equation.
- Krall proved the equivalent of the Bochner result for the fourth-order case.

Legendre-type

- **Differential equation** Let $A \in (0, +\infty)$ be given; then the Lagrange symmetric equation is

$$\left((1-x^2)^2 y''(x)\right)'' - \left(\left[8 + 4A(1-x^2)\right] y'(x)\right)' = \lambda y(x) \qquad (1)$$

for all $x \in (-1, +1)$, with $\lambda \in \mathbb{C}$ as the spectral parameter.

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for all $x \in (-1, +1)$, with $\lambda \in \mathbb{C}$ as the spectral parameter.
- **Measure** Let $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be defined by
$$\hat{\mu} : \begin{array}{ll} = -\frac{1}{2}(A+1) & \text{for all } x \in (-\infty, -1) \\ = \frac{1}{2}Ax & \text{for all } x \in [-1, +1] \\ = +\frac{1}{2}(A+1) & \text{for all } x \in (+1, +\infty). \end{array}$$
Let μ be the non-negative Baire measure generated by $\hat{\mu}$; then μ has compact support $[-1, +1]$ and the associated Lebesgue-Stieltjes space $L^2(\mathbb{R}; \mu) \equiv L^2([-1, +1]; \mu)$ has the inner-product
$$\int_{[-1,+1]} f\overline{g}\, d\mu = \frac{f(-1)\overline{g}(-1)}{2} + \frac{A}{2}\int_{-1}^{+1} f(x)\overline{g}(x)\, dx + \frac{f(+1)\overline{g}(+1)}{2}.$$
Note if $f \in L^2(\mathbb{R}; \mu)$ then the values $f(\pm 1)$ are prescribed in $\mathbb{C}$.

- **Self-adjoint operators** The Lagrange symmetric differential expression on the left of (1) generates a continuum of GKN self-adjoint operators in the Lebesgue space $L^2(-1,+1)$; the expression generates a unique self-adjoint operator T^A_{Le} in the Lebesgue-Stieltjes space $L^2([-1,+1];\mu)$. The operator T^A_{Le} determines the spectral properties of the Legendre-type orthogonal polynomials.

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- **Eigenvalues** The spectrum of T_{Le}^A is simple and discrete with eigenvalues given by

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- **Eigenvectors** The eigenvectors of T_{Le}^A are the Legendre-type orthogonal polynomials $\{P_n^A(x) \text{ :for all } x \in [-1,+1] \text{ and } n \in \mathbb{N}_0\}$ with the explicit representation

$$P_n^A(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!\left(A+\frac{1}{2}n(n-1)+2k\right)x^{n-2k}}{2^n k!(n-k)!(n-2k)!}.$$

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- **Completeness** The set of eigenvectors $\{P_n^A(\cdot) : n \in \mathbb{N}_0\}$ is complete in the space $L^2([-1,+1];\mu)$.

Jacobi-type

- **Differential equation** Let $\alpha \in (-1, +\infty)$ and $A \in (0, +\infty)$ be given; then the Lagrange symmetric differential equation is

$$\left((1-x)^{\alpha+2}(1+x)^2 y''(x)\right)'' -$$
$$(x^2 \exp(-x) y''(x))'' - (((2A+2)x+2)\exp(-x)y'(x))' =$$
$$= \lambda(1-x)^{\alpha} y(x) \text{ for all } x \in (-1,+1), \tag{2}$$

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with $\lambda \in \mathbb{C}$ as the spectral parameter.

- **Measure** Let $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned}\hat{\mu} : \quad &= -[(2^{\alpha+1}-1)A+\alpha+1]\{2(\alpha+1)\}^{-1} && \text{for all } x \in (-\infty,-1) \\ &= \frac{A}{2}\int_0^x (1-t)^{\alpha} dt && \text{for all } [-1,+1] \\ &= A\{2(\alpha+1)\}^{-1} && \text{for all } (+1,+\infty)\end{aligned}$$

Let μ be the non-negative Baire measure generated by $\hat{\mu}$; then μ has compact support $[-1,+1]$ and the associated Lebesgue-Stieltjes space $L^2(\mathbb{R};\mu) \equiv L^2([-1,+1];\mu)$ has the inner-product

- $$\int_{[-1,+1]} f\overline{g}\, d\mu = \frac{f(-1)\overline{g}(-1)}{2} + \frac{A}{2}\int_{-1}^{+1} f(x)\overline{g}(x)(1-x)^{\alpha}\, dx.$$

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- **Eigenvalues** The spectrum of $T_J^{\alpha,A}$ is simple and discrete with eigenvalues given by

  $$\lambda_n^{\alpha,A} = n(n+\alpha+1)(n^2+(\alpha+1)n+4A2^{\alpha}+\alpha) \text{ for all } n \in \mathbb{N}_0.$$

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- **Eigenvectors** The eigenvectors of $T_J^{\alpha,A}$ are the Jacobi-type orthogonal polynomials $\{P_n^{\alpha,A}(x)$:for all $x \in [-1,+1]$ and $n \in \mathbb{N}_0\}$ with the explicit representation

- **Eigenvectors**

$$P_n^{\alpha,A}(x) = \sum_{k}^{n} \frac{(-1)^{n-k}\binom{n}{k}\Gamma(\alpha+n+k+1)}{(k+1)!} \times \\ \times \frac{\left(k(n+\alpha)(n+1)+(k+1)2^{\alpha+1}A\right)}{\Gamma(\alpha+n+1)} \left(\frac{x+1}{2}\right)^k .$$

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- **Completeness** The set of eigenvectors $\{P_n^{\alpha,A}(\cdot) : n \in \mathbb{N}_0\}$ is complete in the space $L^2([-1,+1];\mu)$.

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The completeness of the two sets of Legendre-type and Jacobi-type orthogonal polynomials in their respective Lebesgue-Stieltjes spaces $L^2([-1,+1];\mu)$ may be viewed as due to:

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The completeness of the two sets of Legendre-type and Jacobi-type orthogonal polynomials in their respective Lebesgue-Stieltjes spaces $L^2([-1,+1];\mu)$ may be viewed as due to:

1. The Baire measures μ in the respective cases have compact support $[-1,+1]$ on the real line $\mathbb{R}$.
2. The self-adjoint differential operators T_{Le}^{A} and $T_{J}^{\alpha,A}$ have discrete simple spectra in their respective spaces $L^2([-1,+1];\mu)$ and so, from the spectral theorem, the two sets of eigenvectors are complete in these spaces.

Laguerre-type

- **Differential equation** Let $A \in (0, +\infty)$ be given; then the Lagrange symmetric differential equation is

$$\begin{aligned}(x^2 \exp(-x)y''(x))'' - (((2A+2)x+2)\exp(-x)y'(x))' \\ = \lambda \exp(-x)y(x) \text{ for all } x \in (0, +\infty) \qquad (3)\end{aligned}$$

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- **Measure** Let $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{\mu} : \begin{array}{ll} = -1/A & \text{for all } x \in (-\infty, 0) \\ = 1 - \exp(-x) & \text{for all } x \in [0, +\infty). \end{array}$$

Let μ be the non-negative Baire measure generated by $\hat{\mu}$; then μ has support $[0, +\infty)$ and the associated Lebesgue-Stieltjes space $L^2(\mathbb{R}; \mu) \equiv L^2([0, +\infty); \mu)$ has the inner-product

$$\int_{[0,+\infty)} f\overline{g}\, d\mu = \frac{f(0)\overline{g}(0)}{A} + \int_0^\infty f(x)\overline{g}(x) \exp(-x)\, dx.$$

Note if $f \in L^2(\mathbb{R}; \mu)$ then the values $f(0)$ are prescribed in $\mathbb{C}$.

- **Self-adjoint operators** The Lagrange symmetric differential expression on the left of (3) generates a continuum of GKN self-adjoint operators in the weighted Lebesgue space $L^2((0,+\infty);\exp(-x))$; the expression generates a unique self-adjoint operator T^A_{La} in the Lebesgue-Stieltjes space $L^2([0,+\infty);\mu)$. The operator T^A_{La} determines the spectral properties of the Laguerre-type orthogonal polynomials.

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- **Eigenvalues** The spectrum of T_{La}^A is simple and discrete with eigenvalues given by

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- **Completeness** The set of eigenvectors $\{L_n^A(\cdot) : n \in \mathbb{N}_0\}$ is complete in the space $L^2([0,+\infty);\mu)$.

Bessel functions

- **Differential equation** One form of the classical Bessel differential equation of second-order, when the Bessel parameter $\nu = 0$, is the Sturm-Liouville differential equation

$$-(xy'(x))' = \lambda xy(x) \text{ for all } x \in (0, +\infty), \tag{4}$$

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- **Spectral theory** The spectral theory for this differential equation is considered in the weighted Hilbert function space $L^2((0, +\infty); x)$. All self-adjoint differential operators generated by this differential equation in $L^2((0, +\infty); x)$ have the positive half-line $[0, +\infty) \subset \mathbb{C}$ as continuous spectra.

Bessel-type

- **Differential equation** Let $M \in (0, +\infty)$ be given; then the Lagrange symmetric, Bessel-type differential equation of the fourth-order, is
$$(xy''(x))'' - \left((9x^{-1} + 8M^{-1}x)y'(x)\right)' = \Lambda xy(x) \text{ for all } x \in (0, +\infty),$$
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- **Solutions** Solutions of this equation can be formed from linear combinations of the classical Bessel and modified Bessel functions:

$$(i)\ J_0(x\lambda) \text{ and } J_1(x\lambda)\ (ii)\ Y_0(x\lambda) \text{ and } Y_1(x\lambda)$$
$$(iii)\ K_0\left(x\sqrt{\lambda^2 + 8M^{-1}}\right) \text{ and } K_1\left(x\sqrt{\lambda^2 + 8M^{-1}}\right)$$
$$(iv)\ I_0\left(x\sqrt{\lambda^2 + 8M^{-1}}\right) \text{ and } I_1\left(x\sqrt{\lambda^2 + 8M^{-1}}\right),$$

where $\Lambda \equiv \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1})$ for all $\lambda \in \mathbb{C}$ and all $M > 0$.

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where $\Lambda \equiv \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1})$ for all $\lambda \in \mathbb{C}$ and all $M > 0$.
- **Examples** Let $c = \sqrt{\lambda^2 + 8M^{-1}}$ and $d = 1 + M(\lambda/2)^2$ and define
$$J_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x)$$
$$K_\lambda^{0,M}(x) := dK_0(cx) + \tfrac{1}{2}cMx^{-1}K_1(cx).$$

- **Spectral theory** The spectral theory for this differential equation can be considered in the space $L^2((0,+\infty);x)$. All self-adjoint differential operators generated by this differential equation in $L^2((0,+\infty);x)$ have the positive half-line $[0,+\infty) \subset \mathbb{C}$ as continuous spectra.

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- **Measure** The spectral theory for this differential equation may also be considered (in comparison with the orthogonal polynomial type differential equations) in a Lebesgue-Stieltjes Hilbert function space. Let the parameter $k \in (0,+\infty)$ and define $\hat{\mu}_k : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \hat{\mu}_k(x) : &= -k \text{ for all } x \in (-\infty,0) \\ &= x^2/2 \text{ for all } x \in [0,+\infty). \end{aligned}$$

Let μ be the non-negative Baire measure generated by $\hat{\mu}$; the associated space $L^2([0,+\infty);\mu)$ has the inner-product

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- **Spectral theory** For each $k \in (0,\infty)$ the Bessel-type differential expression generates a unique self-adjoint operator T_B^k in the space $L^2([0,+\infty);\mu)$; the spectrum of T_B^k is continuous on $[0,+\infty) \subset \mathbb{C}$.

Connections

The Bessel-type functions were discovered by Everitt and Markett in 1992 at the SERC meeting held at the Department of Mathematics, University of Cardiff.
We have the following special function connections, which consolidate the position of the Bessel and Bessel-type functions, with respect to the orthogonal and orthogonal-type polynomials.

$$\begin{array}{ccc}
\begin{array}{c}\text{Jacobi polynomials}\\ (1-x)^{\alpha}(1+x)^{\beta}\end{array} & \longrightarrow & \begin{array}{c}\text{Jacobi-type polynomials}\\ (1-x)^{\alpha}\\ +\frac{1}{2}\delta(x+1)\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{c}\text{Laguerre polynomials}\\ x^{\alpha}\exp(-x)\end{array} & \longrightarrow & \begin{array}{c}\text{Laguerre-type polynomials}\\ x^{\alpha}\exp(-x)+\frac{1}{A}\delta(x)\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{c}\text{Bessel functions}\\ x^{2\alpha+1}\end{array} & \longrightarrow & \begin{array}{c}\text{Bessel-type functions}\\ x^{2\alpha+1}+k\delta(x)\end{array}
\end{array}$$

Partial differential equations

On a visit to the University of Birmingham in the summer of 2003 Michael Plum discovered a linear ordinary fourth-order partial differential equation which is connected with the fourth-order Bessel equation.
If the Laplacian ∇^2 partial differential expression in R^2 is written in polar co-ordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

then the Plum equation has the form, with $u = u(r, \theta)$,

$$\nabla^4 u - \gamma \nabla^2 u - \frac{4\gamma}{r^2} u = \Lambda u.$$

Here $\gamma > 0$ is determined by $\gamma = 8M^{-1}$ where $M > 0$ is the parameter in the fourth-order Bessel equation , and $\Lambda \in \mathbb{C}$ is a spectral parameter.

Solutions of this partial differential equation can be obtained by a limited form of the method of separation of variables to give

$$u(r,\theta) = v(r)w(\theta) \text{ for all } r \in (0,\infty) \text{ and } \theta \in [0,2\pi],$$

where:

- The factor $v(\cdot)$ is a solution of the fourth-order Bessel-type differential equation

$$(ry''(r))'' - \left((9r^{-1} + 8M^{-1}r)y'(r)\right)' = \Lambda ry(r) \text{ for all } r \in (0,+\infty).$$

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- The factor $w(\cdot)$ is a solution of the (Sturm-Liouville) differential equation

$$-w''(\theta) = 4w(\theta).$$

  Note that the factor 4 in this second ordinary differential is critical to obtaining the separated property of the partial differential equation. Thus, for some $A, B \in \mathbb{R}$ we have $w(\theta) = A\cos(2\theta) + B\sin(2\theta)$.

Boundary value problems

Some boundary value problems for the Plum partial differential equation may be of interest in applied mathematics.
Possible problems could be concerned with considering solutions of the partial differential equation, satisfying boundary conditions on the following domains of the plane:

1. Bounded domain: $0 < r \leq R$, and $0 \leq \theta \leq \pi$ or $0 \leq \theta \leq 2\pi$

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3. Quarter annulus domain: $0 < a \leq r \leq b < +\infty$ and $0 \leq \theta \leq \pi/2$
4. Plane domain: $0 < r < +\infty$, and $0 \leq \theta \leq 2\pi$.

The biharmonic partial differential equation

If in the Bessel-type equation we take the parameter $\gamma = 0$ then we obtain the fourth-order biharmonic equation in the plane, again with Λ as a spectral parameter

$$\nabla^4 u = \Lambda u;$$

this action is the equivalent to letting the parameter $M \to +\infty$.
The Plum method can be applied to this polar form to obtain separated solutions of the form

$$u(r, \theta) = v(r)w(\theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0.2\pi]$$

where, as before, the factor $w(\cdot)$ satisfies the equation $-w'' = 4w$, but now the factor $v(\cdot)$ has to be a solution of the fourth-order ordinary equation, in Lagrange symmetric form,

$$(rv''(r))'' - (9r^{-1}v'(r))' = \Lambda rv(r) \text{ for all } r \in (0, \infty).$$

This last fourth-order ordinary differential equation can also to be studied in the Hilbert function space $L^2((0, \infty); r)$.